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Exact solutions for Hele-Shaw flows with surface tension: The Schwarz-function approach

Giovani L. Vasconcelos
The James Franck Institute and the Department of Physics, The University of Chicago, 5640 South Ellis Avenue, Chicago, Illinois 60637

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An alternative derivation of the two-parameter family of solutions for a Hele-Shaw flow with surface tension reported previously by Vasconcelos and Kadanoff [Phys. Rev. A **44**, 6490 (1991)] is presented. The method of solution given here is based on the formalism of the Schwarz function: an ordinary differential equation for the Schwartz function of the moving interface is obtained and then solved.

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The primary goal of this paper is to present an alternative derivation of a class of exact solutions for Hele-Shaw flows with surface tension, which was recently reported by Vasconcelos and Kadanoff [1]. These solutions describe an interface with a bubblelike shape moving with a constant velocity in an unbounded Hele-Shaw cell. The interface however does not form a closed curve, and additional boundary conditions ("slip walls") must be introduced for the solutions to be physically meaningful. In spite of this somewhat arbitrary geometry these solutions are nonetheless of interest because they are to date the only known nontrivial explicit solutions to the Hele-Shaw problem with surface tension [2]. One hopes that a better understanding of such solutions might provide insights into constructing exact solutions in more realistic setups, such as the celebrated Saffman-Taylor finger in a channel [3]. The finger solution was first observed in the experiments of Saffman and Taylor [4] over 30 years ago, but an analytical solution to the problem (with surface tension) has not yet been found.

The solutions originally reported in Ref. [1] were obtained in terms of a conformal mapping, $z=H(w)$, that maps the exterior of the unit circle ($|w|\geq 1$) in the auxiliary complex w plane onto the actual fluid region in the z plane. In the conformal-mapping approach, obtaining the correct form of the mapping function is essentially a product of guesswork. Here, on the other hand, I use the formalism of the Schwarz function [5] to give a more systematic (and more elegant) derivation of these solutions. As we shall see below, in this approach, the mapping function is related to the Schwarz function of the moving interface. The exact form of this mapping function is then obtained as a solution to an ordinary differential equation satisfying additional symmetry requirements.

In the past few years, the Schwarz-function method has become a useful analytical tool to treat Hele-Shaw moving boundary problems [6]. In particular, this approach has been successful in constructing several new [7,8] (and old [9]) solutions to the problem. In what follows, I will start with the standard mathematical formulation of two-phase Hele-Shaw flows and briefly recall how one recasts this problem in terms of the Schwarz function of the moving interface. I will then proceed to solve the problem in the case of the geometry studied in Ref. [1].

We consider the problem of a bubble moving with constant velocity U along the x direction in a Hele-Shaw cell. The fluid inside the bubble has a negligible viscosity and is kept at constant pressure. The fluid outside the bubble has a larger viscosity and is incompressible. The velocity $\mathbf{v}(x,y)$ in the (viscous) fluid is given by Darcy's law:

$$\mathbf{v}=-\frac{b^2}{12\mu}\nabla p=\nabla\phi \ , \tag{1}$$

where b is the cell gap, μ is the viscosity, p is the pressure, and ϕ is the so-called velocity potential. As is standard in two-dimensional hydrodynamics, we introduce the complex potential $W(z)=\phi(x,y)+i\psi(x,y)$, where $z=x+iy$ and ψ is the stream function. The appropriate complex potential in the frame moving with the bubble is $\Phi=W-Uz$. Let $\mathcal{D}$ denote the region occupied by the fluid, and $\mathcal{C}$ the fluid-bubble interface. Then $\Phi(z)$ must be an analytic function in the domain $\mathcal{D}$ and satisfy the following boundary conditions. The imaginary part of Φ on $\mathcal{C}$ must be constant (chosen to be zero) to ensure that the interface is a streamline of the flow. On the other hand, the real part of Φ on $\mathcal{C}$ is given by the jump in pressure across the interface (the Gibbs-Thomson relation), so that

$$\mathrm{Re}\,\Phi=\frac{b^2}{12\mu}\tau\kappa-Ux+\phi_0 \quad \text{on } \mathcal{C} \ , \tag{2}$$

where κ is the curvature of the interface and τ is the surface tension. Here $\phi_0=(b^2/12\mu)p_0$, with p_0 being the pressure inside the bubble. We assume furthermore that the fluid velocity at infinity is a constant V in the x direction. This implies that

$$\Phi\approx(V-U)z \quad \text{as } |z|\to\infty \ . \tag{3}$$

Now we reformulate the problem in terms of the Schwarz function of the interface. This function is defined as follows [5]: Suppose the curve $\mathcal{C}$ is given by the relation $F(x,y)=0$ for some function $F(x,y)$; the Schwarz function of $\mathcal{C}$, denoted by $S(z)$, is obtained by solving the equation $F((z+\bar{z})/2,(z-\bar{z})/2i)=0$ for $\bar{z}$ in the form $\bar{z}=S(z)$. Geometrical properties of a curve can be expressed in terms of its Schwarz function. For instance, the angle θ that the tangent to $\mathcal{C}$ at the point z_0 makes with the real axis is given by [5]

$$S'(z_0)=e^{-i2\theta}, \quad z_0\in\mathscr{C} \ , \tag{4}$$

where the prime indicates differentiation. The curvature κ, on the other hand, can be written as [5]

$$\kappa=\frac{i}{2}\frac{S''}{(S')^{3/2}}, \quad z\in\mathscr{C} \ . \tag{5}$$

Using Eq. (5) and the fact that $x=[z+S(z)]/2$ on $\mathscr{C}$, it then follows from analytic continuation of Eq. (2) that the complex potential can be written as

$$\Phi(z)=\frac{ib^2\tau}{24\mu}\frac{S''(z)}{[S'(z)]^{3/2}}-\frac{U}{2}[z+S(z)]+\phi_0 \ . \tag{6}$$

The following properties of the Schwarz function will be of use later. Suppose that the unit circle in the ζ plane is mapped onto curve $\mathscr{C}$ in the z plane by means of the analytic function $z=f(\zeta)$, which is one-to-one conformal in a neighborhood of $\mathscr{C}$. Then [5]

$$S(z)=\bar{f}\left[\frac{1}{\zeta}\right]=\bar{f}\left[\frac{1}{f^{-1}(z)}\right] \ . \tag{7}$$

Here $\bar{f}$ is the so-called conjugate function of f and is defined as $\bar{f}(z)=\overline{f(\bar{z})}$, where the overbar (on the right-hand side) stands for the complex conjugation. Hereafter we will assume that $\mathscr{C}$ is symmetrical with respect to the x axis, so that $\bar{f}=f$. We assume furthermore that $S(z)$ is analytic in the fluid domain $\mathscr{D}$, so that the curvature term in Eq. (6) is also analytic in $\mathscr{D}$. We then note that if κ does not change sign along $\mathscr{C}$, i.e., the angle θ is a monotonic function of arclength, then according to Eq. (4) the function

$$\zeta=g(z)=\frac{1}{\sqrt{-S'(z)}} \tag{8}$$

maps $\mathscr{C}$ one-to-one onto the unit circle ($|\zeta|=1$) and $\mathscr{D}$ conformally onto the exterior of the unit circle ($|\zeta|>1$). Now consider the inverse mapping

$$z=h(\zeta)=g^{-1}(\zeta) \ . \tag{9}$$

It then follows form Eqs. (7) and (8) that $h(\zeta)$ must satisfy the following compatibility condition:

$$h'\left[\frac{1}{\zeta}\right]=h'(\zeta) \ . \tag{10}$$

We now seek solutions for the complex potential Φ via the ansatz

$$\Phi(z)=(V-U)[z+S(z)] \ , \tag{11}$$

so that the interface $\mathscr{C}$ is, by construction, a streamline of the flow. We also require that the Schwarz function $S(z)$ decay to a constant at infinity in order to satisfy Eq. (3). According to Eq. (7) the solution for Φ in terms of the mapping function $h(\zeta)$ reads

$$\Phi=(V-U)\left[h(\zeta)+h\left[\frac{1}{\zeta}\right]\right] \ . \tag{12}$$

In order to solve for $h(\zeta)$ we first insert Eq. (11) into Eq. (6). After some simplification, we obtain the following ordinary differential equation for S:

$$z+S-2L^2[(-S')^{-1/2}]'=\frac{2L^2}{R_0} \ , \tag{13}$$

where $L=[\tau b^2/12\mu(2V-U)]^{1/2}$ and $R_0=\tau/p_0$ both have dimensions of length. Here we assume that $2V>U$, so that $0<L<\infty$. (Note that if $U=2V$, one recovers the circular bubble of radius R_0 [2].) Differentiating Eq. (13) twice yields

$$S''-2L^2[(-S')^{-12}]'''=0 \ . \tag{14}$$

In view of Eq. (8), this can be expressed in terms of the function $g(z)$. One then finds

$$g'-L^2g^3g'''=0 \ . \tag{15}$$

Equation (15) can now be easily integrated to give implicit solutions for g in the form

$$z=L\int\left[\frac{g}{Ag^2+Bg+1}\right]^{1/2}dg+C \ , \tag{16}$$

where A, B, and C are constants of integration. In order to satisfy the compatibility condition (10) we must have $A=1$. We fix C with an appropriate choice of the origin and write the final solution for h as

$$h(\zeta)=L\int_{-1}^{\zeta}\left[\frac{\zeta}{\zeta^2+2a\zeta+1}\right]^{1/2}d\zeta \ , \tag{17}$$

where a is real parameter, so that $\bar{h}=h$. Here we take $a>1$, in which case the singularities of the integrand lie on the negative real semiaxis. The corresponding "physical domain" Ω in the ζ plane is shown in Fig. 1. It will be understood that at the lower limit, $\zeta=-1$, the square root in the integrand takes the positive value and that it is then varied continuously along a contour lying entirely in Ω to the upper limit ζ. We shall note that Eq. (17) corresponds exactly to the solution reported earlier by Vasconcelos and Kadanoff [1]. [More precisely, their mapping function $H(w)$, see Eq. (5) of Ref. [1], is given by $H(w)=-ih(-w^2)$, with h as in Eq. (17) above.]

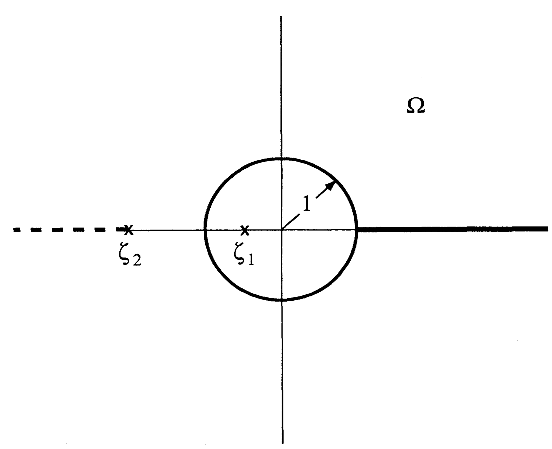

FIG. 1. The physical domain Ω in the ζ plane. Here ζ_1 and ζ_2 are the singularities of the integrand in Eq. (17). The branch cut at $\zeta<\zeta_2$ corresponds to a constant-pressure inlet and the cut at $\zeta>1$ to slip walls (see Fig. 2).

It is perhaps worth mentioning that Eq. (13) can be viewed as a "complexified" version of the meniscus equation for equilibrium capillary surfaces [10]. In fact, on $\mathcal{C}$, Eq. (13) reduces to the standard meniscus equation $\kappa = x/L^2$, with "gravity" acting along the x axis and L playing the role of the capillary length [10]. The solutions given above correspond to two-dimensional sessile drops [11]. This is illustrated in Fig. 2, where the solution for $a = 1.1$ is shown. Note, as already mentioned, the presence of additional boundary conditions: slip walls to which the bubbles are attached and a constant-pressure inlet [1].

In closing, I would like to point out that it is not surprising that the interface in our solutions does not form a closed curve. For, as Millar [9] has shown, the circle is the *only* solution for a single, closed bubble in an unbounded Hele-Shaw cell with surface tension for which the Schwarz function is analytic in the fluid domain, i.e., outside the bubble. It remains an open question whether there exists a solution for which the Schwarz function possesses singularities in the fluid region but the complex potential is nevertheless analytic.

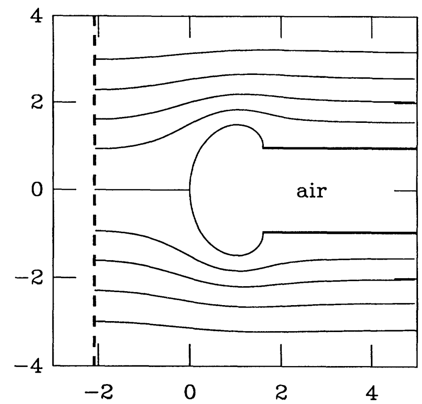

FIG. 2. The bubble solution (in the moving frame) for the case $a = 1.1$ and $L = 1$. The dashed line indicates the constant-presure inlet and the thick solid lines indicate the slip walls [1]. Also shown are some streamlines of the flow.

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